RP334

INTERPOLATION OF THE O. S. A. "EXCITATION" DATA BY THE FIFTH-DIFFERENCE OSCULATORY FORMULA

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ABSTRACT

In order to compute the dominant wave length and purity of a color stimulus by means of the O. S. A. "excitation" data, two values must be obtained by interpolation. The adoption of the osculatory formula for this interpolation permits the computations to be made with perfect reproducibility. Each of the O. S. A. curves by this method is represented as a series of parabolas of the fifth degree which join at the values specified at every 10 mμ so as to have a common slope and curvature at the junction point. Interpolated values have been computed according to this formula for every millimicron.

Since 1922, colorimetric computation in America has been carried out mostly by means of the O. S. A. "excitation" data[1] in their extrapolated form.[2] In spite of the fact that these data are based on rather meager experimental measurements they represent the characteristics of the average normal eye with a satisfactory degree of approximation, and they have been used as the basis for extensive work in color standardization. In computing dominant wave length and colorimetric purity[3] by means of these data interpolated values are required, and have commonly been found from an interpolation graph.

It is not to be supposed, of course, that any form of interpolation, graphical or by formula, yields any information concerning the true course of the O. S. A. "excitation curves" between the specified points. Theoretically that course is wholly unspecified, and there are a large number of equally reliable curves to be drawn through the intervals. As in the case of the standard visibility function[4] it has been found convenient in the work of the National Bureau of Standards to adopt arbitrarily a single set of curves. The interpolated values adopted are found by a method superior to any graphical interpolation because the values may be reproduced at any time anywhere, and it is believed that the solution by fifth-difference, osculatory interpolation is more suitable than solutions by other formulas for interpolation because it combines continuity in function, in first derivative and in second derivative with considerable com-

[1] L. T. Troland, Report of Committee on Colorimetry for 1920–21, J. Opt. Soc. Am. and Rev. Sci. Inst., **6**, pp. 547–553; 1922.

[2] Spectrophotometry, Report of O. S. A. Progress Committee for 1922–23, J. Opt. Soc. Am. and Rev. Sci. Inst., **10**, p. 230; 1925.

[3] I. G. Priest, The Computation of Colorimetric Purity, J. Opt. Soc. Am. and Rev. Sci. Inst., **9**, pp. 503–520; 1924. D. B. Judd, The Computation of Colorimetric Purity, J. Opt. Soc. Am. and Rev. Sci. Inst., **13**, pp. 133–152; 1926.

[4] D. B. Judd, Extension of the Standard Visibility Function to Intervals of 1 Millimicron by Third-difference Osculatory Interpolation, B. S. Jour. Research, **6**, pp. 465–471; 1931.

putational convenience. Indeed, the ease of applying this method is so great that the labor involved is not much greater than that of the graphical method.

If λ be the wave length in millimicrons, λ_0, the value of the wave length at the beginning of the 10 mμ interval within which $f(\lambda-\lambda_0)$ is to be defined by interpolation, and if $\Delta_1 f(-20), \ldots \Delta_5 f(-20)$ be the five leading major differences, then the fifth-difference, osculatory interpolation formula may be written:

$$f(\lambda-\lambda_0)=f(-20)+k_1\Delta_1 f(-20)+k_2\Delta_2 f(-20)+k_3\Delta_3 f(-20)$$
$$+k_4\Delta_4 f(-20)+k_5\Delta_5 f(-20),\ \lambda_0<\lambda<\lambda_0+10$$

where the coefficients, k_1 to k_5, applied to the leading major differences are defined:

$$k_1\equiv(\lambda-\lambda_0+20)/10$$
$$k_2\equiv(\lambda-\lambda_0+20)\ (\lambda-\lambda_0+10)/200$$
$$k_3\equiv(\lambda-\lambda_0+20)\ (\lambda-\lambda_0+10)\ (\lambda-\lambda_0)/6{,}000$$
$$k_4\equiv(\lambda-\lambda_0+20)\ (\lambda-\lambda_0+10)\ (\lambda-\lambda_0)\ (\lambda-\lambda_0-10)/240{,}000$$
$$k_5\equiv(\lambda-\lambda_0)^3\ (\lambda-\lambda_0-10)\ (5\lambda-5\lambda_0-70)/2{,}400{,}000$$

This formula is another form of that derived in 1880 by Sprague.[5] Sprague's formula was restated in nearly the form used above by Karup [6] 20 years later who used, however, three of the central differences in place of three of the leading differences which we have used here with appropriate changes in the coefficients of these differences. Since this formula, applied interval by interval, yields a series of parabolas which join at the specified points so as to have common slopes and osculating circles at the junction points, it was called by Karup a formula for osculatory interpolation. Osculatory interpolation has attracted some attention [7] since its discovery, and, indeed, has been extensively used in the construction of the United States Life Tables.[8] Probably the best general treatment of osculatory interpolation is due to Glover.[9]

Although the formulas for osculatory interpolation have usually been applied in the past by computing the leading minor differences (that is, values which would be obtained by differencing the desired, evenly spaced, interpolated values) from the leading major differences, and then deriving the desired interpolated values by continuous addition, the interpolation of the O. S. A. "excitation" data has been accomplished by finding for each value the products indicated in the formula and actually taking their sum. With a computing

[5] T. B. Sprague, Explanation of a New Formula for Interpolation, J. Inst. Actuaries, **22**, pp. 270-285; 1880.

[6] J. Karup, On a New Mechanical Method of Graduation, Trans. Second International Actuarial Congress, p. 82; 1899.

[7] George King, On the Construction of Mortality Tables from Census Returns and Records of Deaths, J. Inst. Actuaries, **42**, pp. 238-246; 1908. James Buchanan, Osculatory Interpolation by Central Differences; with an Application to Life Table Construction, J. Inst. Actuaries, **42**, pp. 369-394; 1908. See also an appendix by G. J. Lidstone, Alternative Demonstration of the Formula for Osculatory Interpolation, pp. 394-397. George King, On a New Method of Constructing and of Graduating Mortality and Other Tables, J. Inst. Actuaries, **43**, pp. 109-184; 1909

[8] J. W. Glover, United States Life Tables, 1890, 1901, 1910, and 1901-1910, pp. 344-347, 372-388; 1921.

[9] J. W. Glover, Derivation of the United States Mortality Table by Osculatory Interpolation, Quarterly Publications of the American Statistical Association, **12**, pp. 87-93; 1910.

machine it was found possible to obtain nine interpolated values and to check them by an independent method in about 20 minutes;[10] it seems doubtful whether the continuous addition method would be much more expeditious.

We take, then, $\lambda-\lambda_0$ equal in succession to 1, 2 . . . 9, and $f(\lambda-\lambda_0)$ interval by interval equal to ρ_0, γ_0 and β_0 in succession, where ρ_0, γ_0, and β_0 are the O. S. A. "excitation curves" as extrapolated by Priest and Gibson.[11]

Table 1 shows by an example how to compute the leading, descending, major differences. We have taken $f(\lambda-\lambda_0)$ equal to ρ_0, and λ_0 equal to 580 mμ. The leading, descending, major differences appear in the first row.

Table 2 shows by the same example the computation of the leading ascending, major differences which were used in an independent check on the computation; the leading, ascending, major differences appear in the first row of this table.

Table 3 gives the values of the coefficients, k_1 to k_5, for $\lambda-\lambda_0$ equal to 1, 2, . . . 9.

Table 4 shows the details of the computation of the interpolated values according to the fifth-difference formula for the same interval and function referred to in Tables 1 and 2. The check by means of the ascending differences is also given for this interval and function.

Table 5 gives the results of these computations for ρ_0, γ_0, and β_0.[12] Since there was some rejection error in writing down the products (for example, the coefficients, k_1 to k_5, given in Table 3 were taken to the nearest fourth decimal), the values given in Table 5 are not quite what would be found by rigorous evaluation according to the formula for fifth-difference osculatory interpolation; they may be taken as correct, however, to within 3 in the second decimal. The values of Table 5 were further checked for clerical errors by computing the first and second minor differences.

TABLE 1.—*Computation of the leading descending major differences,* $\lambda_0=580$ mμ

$\lambda-\lambda_0$ in mμ	λ in mμ	$f(\lambda-\lambda_0)=\rho_0$	$\Delta_1\rho_0$	$\Delta_2\rho_0$	$\Delta_3\rho_0$	$\Delta_4\rho_0$	$\Delta_5\rho_0$
−20	560	466	+39	−24	+24	−64	+121
−10	570	505	+15	0	−40	+57	
0	580	520	+15	−40	+17		
10	590	535	−25	−23			
20	600	510	−48				
30	610	462					

[10] The check was carried out by taking the differences in the ascending order rather than in the descending order as indicated in the formula. See Tables 2 and 4. It might naturally be supposed that about twice the time to calculate nine values would be required to calculate and check them by an independent method, but this is not quite the case. The time actually required is considerably less than twice because the products found for checking the values in one interval may be used to calculate values in the four subsequent intervals.

[11] See footnote 2, p. 85. The values referred to here are included in Table 5 along with the values obtained from them by interpolation.

[12] The values for ρ_0 from 451 to 459 and from 461 to 469 mμ result from substituting in the formula $f(-20)$ equal to 4 and 1, respectively; that is, we have taken ρ_0 for 430 and 440 mμ equal to ρ_0 for 470 and 460 mμ, respectively, instead of zero as shown in Table 5. This choice was made in order to bring the interpolated function to zero at 450 mμ with a zero slope; then, for λ less than 450 mμ, ρ_0 is arbitrarily set at zero instead of at the values which would be obtained by mechanical application of the formula. Similarly for β_0 between 590 and 610 mμ, we choose β_0 in the formula as 1 and 2 for 620 and 630, respectively, although β_0 is given in Table 5 as zero for wave lengths greater than 610 mμ.

TABLE 2.—*Computation of the leading ascending major differences,* $\lambda_0 = 580\ m\mu$

$\lambda-\lambda_0$ in $m\mu$	λ in $m\mu$	$f(\lambda-\lambda_0) = p_0$	$\nabla_1 p_0$	$\nabla_2 p_0$	$\nabla_3 p_0$	$\nabla_4 p_0$	$\nabla_5 p_0$
30	610	462	+48	−23	−17	+57	−121
20	600	510	+25	−40	+40	−64	
10	590	535	−15	0	−24		
0	580	520	−15	−24			
−10	570	505	−39				
−20	560	466					

TABLE 3.—*Coefficients,* k_1 *to* k_5*, for interpolation to tenths by fifth-difference osculatory interpolation*

$\lambda-\lambda_0$	k_1	k_2	k_3	k_4	k_5
1	+2.1	+1.155	+0.0385	−0.0086625	+0.00024375
2	+2.2	+1.320	+.0880	−.0176000	+.00160000
3	+2.3	+1.495	+.1495	−.0261625	+.00433125
4	+2.4	+1.680	+.2240	−.0336000	+.00800000
5	+2.5	+1.875	+.3125	−.0390625	+.01171875
6	+2.6	+2.080	+.4160	−.0416000	+.01440000
7	+2.7	+2.295	+.5355	−.0401625	+.01500625
8	+2.8	+2.520	+.6720	−.0336000	+.01280000
9	+2.9	+2.755	+.8265	−.0206625	+.00759375

TABLE 4.—*Example of interpolation to tenths by the fifth-difference osculatory formula, descending differences; check by ascending differences*

Take $\lambda_0=580\ m\mu$, and f $(\lambda-\lambda_0=p_0$; then from Table 1:

$$f\ (\lambda-\lambda_0)=466+39k_1-24k_2+24k_3-64k_4+121k_5$$

The coefficients, k_1 *to* k_5 may be found in Table 3.

λ in $m\mu$	$\lambda-\lambda_0$	$+39k_1$	$-24k_2$	$+24k_3$	$-64k_4$	$+121k_5$	$f\ (\lambda-\lambda_0)$
581	1	+81.90	−27.72	+0.92	+0.56	+0.02	521.68
582	2	+85.80	−31.68	+2.11	+1.13	+.19	523.55
583	3	+89.70	−35.88	+3.59	+1.68	+.52	525.61
584	4	+93.60	−40.32	+5.38	+2.15	+.97	527.78
585	5	+97.50	−45.00	+7.50	+2.50	+1.42	529.92
586	6	+101.40	−49.92	+9.98	+2.66	+1.74	531.86
587	7	+105.30	−55.08	+12.85	+2.57	+1.82	533.46
588	8	+109.20	−60.48	+16.13	+2.15	+1.55	534.55
589	9	+113.10	−66.12	+19.84	+1.32	+.92	535.06

Check by ascending differences: From Table 2 we may write:

$$f\ (\lambda-\lambda_0)=462+48k_1'-23k_2'-17k_3'+57k_4'-121k_5'$$

The coefficients, k_1' to k_5', may be found in Table 3 by reading the values of the coefficients, k_1 to k_5, for $10-\lambda+\lambda_0$ instead of for $\lambda-\lambda_0$.

λ in $m\mu$	$\lambda-\lambda_0$	$+48k_1'$	$-23\ k_2'$	$-17\ k_3'$	$+57\ k_4'$	$-121\ k_5'$	$f\ (\lambda-\lambda_0)$
581	1	+139.20	−63.36	−14.05	−1.18	−0.92	521.69
582	2	+134.40	−57.96	−11.42	−1.92	−1.55	523.55
583	3	+129.60	−52.79	−9.10	−2.29	−1.82	525.60
584	4	+124.80	−47.84	−7.07	−2.37	−1.74	527.78
585	5	+120.00	−43.13	−5.31	−2.23	−1.42	529.91
586	6	+115.20	−38.64	−3.81	−1.92	−.97	531.86
587	7	+110.40	−34.39	−2.54	−1.49	−.52	533.46
588	8	+105.60	−30.36	−1.50	−1.00	−.19	534.55
589	9	+100.80	−25.56	−.65	−.50	−.02	535.07

TABLE 5.—*The O. S. A. "excitation" data extended to values for every millimicron by fifth-difference osculatory interpolation*

Original values appear in bold-face type

λ in mμ	ρ_0	γ_0	β_0	λ in mμ	ρ_0	γ_0	β_0	λ in mμ	ρ_0	γ_0	β_0
410	**0**	**0**	**433**	**470**	**4**	**81**	**697**	**530**	**307**	**572**	**43**
1	0.00	0.05	450.34	1	4.55	85.30	678.06	1	314.02	576.22	41.46
2	.00	.11	466.85	2	5.16	89.54	658.31	2	320.98	580.13	39.96
3	.00	.19	482.53	3	5.83	93.74	637.78	3	327.91	583.77	38.48
4	.00	.27	497.65	4	6.57	97.88	616.48	4	334.76	587.14	37.03
5	.00	.37	512.88	5	7.41	101.97	594.40	5	341.50	590.26	35.62
6	.00	.49	528.98	6	8.39	106.01	571.56	6	348.14	593.18	34.24
7	.00	.61	546.72	7	9.50	110.01	547.94	7	354.62	595.90	32.88
8	.00	.73	566.71	8	10.81	114.01	523.60	8	360.94	598.44	31.56
9	.00	.86	589.18	9	12.31	118.00	498.58	9	367.06	600.81	30.26
420	**0**	**1**	**614**	**480**	**14**	**122**	**473**	**540**	**373**	**603**	**29**
1	0.00	1.15	640.68	1	15.88	126.02	446.93	1	378.78	605.02	27.76
2	.00	1.30	669.70	2	17.97	130.02	420.08	2	384.38	606.91	26.54
3	.00	1.47	701.04	3	20.28	134.01	392.56	3	389.80	608.68	25.36
4	.00	1.66	734.21	4	22.80	138.04	364.62	4	395.05	610.28	24.20
5	.00	1.85	768.20	5	25.50	142.24	336.87	5	400.15	611.64	23.06
6	.00	2.05	801.90	6	28.38	146.68	309.88	6	405.12	612.68	21.96
7	.00	2.28	834.20	7	31.38	151.50	284.32	7	409.98	613.29	20.90
8	.00	2.50	864.09	8	34.48	156.81	260.67	8	414.73	613.40	19.90
9	.00	2.74	891.04	9	37.69	162.64	239.24	9	419.40	612.97	18.92
430	**0**	**3**	**915**	**490**	**41**	**169**	**220**	**550**	**424**	**612**	**18**
1	0.00	3.28	936.48	1	44.44	175.86	202.72	1	428.50	610.54	17.12
2	.00	3.58	955.33	2	47.97	183.19	187.64	2	432.89	608.56	16.28
3	.00	3.91	971.38	3	51.62	191.02	174.83	3	437.16	606.01	15.48
4	.00	4.26	984.68	4	55.39	199.36	164.12	4	441.33	602.96	14.74
5	.00	4.64	995.42	5	59.32	208.20	155.18	5	445.44	599.46	14.02
6	.00	5.06	1003.81	6	63.47	217.55	147.56	6	449.51	595.60	13.34
7	.00	5.50	1010.18	7	67.88	227.42	140.83	7	453.59	591.47	12.72
8	.00	5.96	1014.76	8	72.60	237.79	134.60	8	457.69	587.15	12.12
9	.00	6.47	1017.68	9	77.64	248.66	128.66	9	461.83	582.66	11.54
440	**0**	**7**	**1019**	**500**	**83**	**260**	**123**	**560**	**466**	**578**	**11**
1	0.00	7.59	1018.48	1	88.64	271.80	117.84	1	470.18	573.13	10.49
2	.00	8.21	1015.95	2	94.59	284.06	113.19	2	474.44	568.08	10.01
3	.00	8.89	1011.45	3	100.82	296.78	108.98	3	478.76	562.88	9.57
4	.00	9.62	1005.18	4	107.34	309.88	105.16	4	483.11	557.50	9.16
5	.00	10.44	997.49	5	114.12	323.27	101.64	5	487.40	551.84	8.76
6	.00	11.33	988.76	6	121.13	336.84	98.40	6	491.54	545.84	8.38
7	.00	12.32	979.38	7	128.33	350.49	95.36	7	495.42	539.41	8.03
8	.00	13.43	969.69	8	135.74	364.11	92.46	8	498.98	532.47	7.68
9	.00	14.65	959.88	9	143.30	377.62	89.69	9	502.17	525.00	7.34
450	**0**	**16**	**950**	**510**	**151**	**391**	**87**	**570**	**505**	**517**	**7**
1	0.01	17.48	939.91	1	158.86	404.25	84.36	1	507.51	508.54	6.67
2	.05	19.06	929.69	2	166.89	417.40	81.69	2	509.64	499.62	6.34
3	.11	20.74	919.44	3	175.08	430.43	78.97	3	511.33	490.24	6.02
4	.20	22.54	909.12	4	183.38	443.22	76.20	4	512.68	480.41	5.71
5	.31	24.52	898.68	5	191.76	455.73	73.42	5	513.78	470.20	5.40
6	.43	26.68	888.06	6	200.16	467.80	70.70	6	514.78	459.67	5.10
7	.56	29.10	877.15	7	208.52	479.33	68.06	7	515.82	448.82	4.80
8	.70	31.78	865.86	8	216.80	490.24	65.55	8	517.02	437.74	4.52
9	.85	34.76	854.14	9	224.96	500.47	63.20	9	518.42	426.46	4.26
460	**1**	**38**	**842**	**520**	**233**	**510**	**61**	**580**	**520**	**415**	**4**
1	1.17	41.51	829.55	1	240.92	518.87	58.89	1	521.68	403.36	3.76
2	1.35	45.31	816.89	2	248.70	527.01	56.87	2	523.55	391.50	3.52
3	1.56	49.42	804.02	3	256.34	534.39	54.92	3	525.60	379.46	3.30
4	1.79	53.77	790.90	4	263.82	541.08	53.06	4	527.78	367.30	3.07
5	2.06	58.32	777.31	5	271.18	547.13	51.26	5	529.92	355.07	2.86
6	2.35	62.95	763.10	6	278.44	552.70	49.52	6	531.86	342.88	2.66
7	2.69	67.60	748.08	7	285.63	557.89	47.84	7	533.46	330.83	2.48
8	3.07	72.17	732.08	8	292.78	562.80	46.19	8	534.55	318.97	2.31
9	3.51	76.64	715.05	9	299.92	567.51	44.58	9	535.06	307.36	2.15

TABLE 5.—*The O. S. A. "excitation" data extended to values for every millimicron by fifth-difference osculatory interpolation*—Continued

λ in mμ	ρ_0	γ_0	β_0	λ in mμ	ρ_0	γ_0	β_0	λ in mμ	ρ_0	γ_0	β_0
590	**535**	**296**	**2**	**620**	**375**	**59**	**0**	**650**	**118**	**3**	**0**
1	534.44	284.89	1.87	1	365.85	55.08	0.00	1	111.72	2.67	0.00
2	533.34	274.08	1.75	2	356.74	51.42	0.00	2	105.74	2.37	0.00
3	531.64	263.58	1.65	3	347.73	48.01	0.00	3	100.08	2.12	0.00
4	529.40	253.40	1.55	4	338.79	44.84	0.00	4	94.72	1.90	0.00
5	526.72	243.47	1.46	5	329.92	41.88	0.00	5	89.64	1.70	0.00
6	523.69	233.74	1.38	6	321.05	39.09	0.00	6	84.82	1.54	0.00
7	520.44	224.18	1.30	7	312.16	36.43	0.00	7	80.28	1.40	0.00
8	517.05	214.70	1.20	8	303.17	33.87	0.00	8	75.96	1.25	0.00
9	513.58	205.30	1.10	9	294.12	31.40	0.00	9	71.87	1.12	0.00
600	**510**	**196**	**1**	**630**	**285**	**29**	**0**	**660**	**68**	**1**	**0**
1	506.26	186.84	0.89	1	275.86	26.69	0.00	1	64.36	0.89	0.00
2	502.42	177.80	.77	2	266.72	24.45	0.00	2	60.95	.77	0.00
3	498.47	168.90	.64	3	257.57	22.28	0.00	3	57.78	.66	0.00
4	494.40	160.16	.50	4	248.42	20.16	0.00	4	54.82	.53	0.00
5	490.10	151.58	.37	5	239.32	18.14	0.00	5	52.04	.41	0.00
6	485.46	143.26	.24	6	230.26	16.22	0.00	6	49.43	.30	0.00
7	480.41	135.21	.14	7	221.28	14.43	0.00	7	46.93	.20	0.00
8	474.82	127.47	.06	8	212.40	12.80	0.00	8	44.55	.12	0.00
9	468.68	120.07	.02	9	203.64	11.32	0.00	9	42.24	.05	0.00
610	**462**	**113**	**0**	**640**	**195**	**10**	**0**	**670**	**40**	**0**	**0**
1	454.85	106.24	0.00	1	186.49	8.82	0.00				
2	447.18	99.81	0.00	2	178.09	7.77	0.00				
3	439.00	93.69	0.00	3	169.83	6.86	0.00				
4	430.34	87.89	0.00	4	161.72	6.06	0.00				
5	421.35	82.39	0.00	5	153.78	5.38	0.00				
6	412.12	77.19	0.00	6	146.08	4.79	0.00				
7	402.78	72.26	0.00	7	138.62	4.26	0.00				
8	393.45	67.59	0.00	8	131.44	3.80	0.00				
9	384.19	63.18	0.00	9	124.57	3.38	0.00				

Figure 1 illustrates the smoothness of the interpolated values obtained by this method; it shows a portion of the ρ_o curve near its maximum which possesses an irregularity of such nature that graphical interpolation is difficult.

It is to be noted that the interpolation has not been carried out beyond the limits 410 and 670 mμ; this serves to emphasize that the present plan is to refrain from using the interpolated values in obtaining the trilinear coordinates corresponding to a given spectral distribution of energy. There would now seem to be little need to make use of intervals smaller than 10 mμ for this purpose although advances in colorimetric technique may be made in the future which will require smaller intervals to be used. The interpolation of the O. S. A. curves by somewhat refined method does not presuppose this advanced technique; it is merely a question of adopting arbitrarily some one of the many equally reliable interpolations so that computed results via the O. S. A. "excitation" data which depend on interpolation may be perfectly intercomparable; the specific purpose, as previously stated, is to make possible the computation of rigorously comparable values of dominant wave length and colorimetric purity from given values of trilinear coordinates. The interpolated results given in Table 5 are sufficient to permit reproducible interpolation for dominant wave lengths to tenths of a millimicron merely from a table of values. If, for special purposes, a comparison of values to hundredths of a millimicron be desired it is probable that an inter-

polation graph based on the values of Table 5 would afford an adequate basis. It may be pointed out, too, that interpolations of any

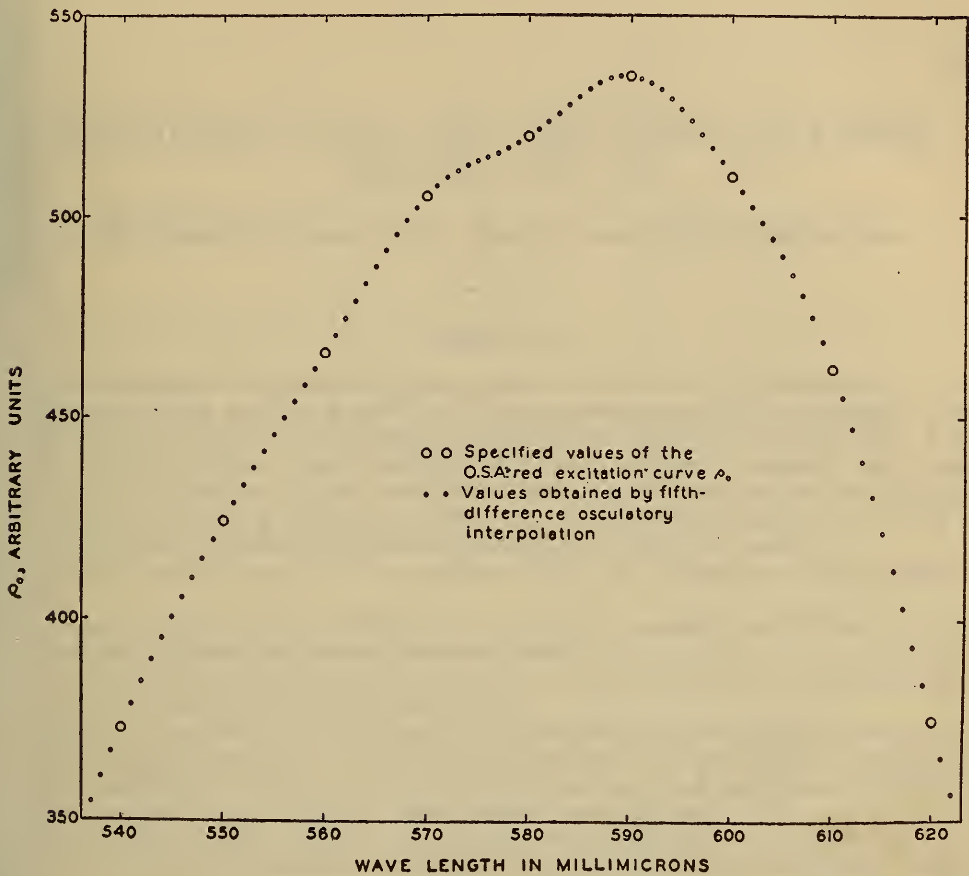

FIGURE 1.—*Example of interpolation of the O. S. A. "excitation" data by the fifth-difference, osculatory formula*

A portion of the red curve near its maximum is chosen for a demonstration of the smoothness of the interpolated values.

precision desired, however great, may be accomplished by carrying the appropriate number of decimal places in computing by the formula.

WASHINGTON, May 12, 1931.